THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4230 2024-25
Lecture 25 (Last Lesson)
April 16, 2025 (Wednesday)

Financial Terminology

Introduction

- **Return:** $S_1^k - S_0^k$
- **Relative Return:** $X_k := \dfrac{S_1^k - S_0^k}{S_0^k}$, set $X = (X_1, \dots, X_n)$ and $m := \mathbb{E}[X],\ \Sigma := \mathrm{Var}[X]$.
- **Bank account with interest rate**: $r > 0$

Now, we define two new terminology as

- w_k: ratio of amount invested in S^k
- w_0: ratio of amount in bank account
- Total return: $w_0 r + w^T X$ so that $\mathbb{E}[R] = w_0 r + m^T w$ and $\mathrm{Var}[R] = w^T \Sigma w$.

Risk-minimizing Problem

Consider the problem:

$$\min_{\substack{w_0 + \sum_{k=1}^n w_k = 1 \\ w_0 r + m^T w \geq \mu}} \frac{1}{2} w^T \Sigma w \tag{P}$$

Under the following conditions:

1. There exists $w_0, w \in \mathbb{R}^n$ such that $w_0 r + m^T w > \mu$
2. $\Sigma \succ 0$

Solution.

- **Existence of optimal solution** w^* is true since $w \mapsto \dfrac{1}{2} w^T \Sigma w$ is coercive.
- Under Slater condition, the Qualification condition holds.
- Define $f(w_0, w) = \dfrac{1}{2} w^T \Sigma w$, $h(w_0, w) = w_0 + \mathbb{1}^T w - 1$ and $g(w_0, w) = \mu - w_0 r - m^T w$.
- By the KKT theorem, there exists $\lambda \geq 0, \gamma \in \mathbb{R}$ such that

$$\begin{cases} \nabla f(w_0^*, w^*) + \lambda \nabla g(w_0^*, w^*) + \gamma \nabla h(w_0^*, w^*) = \mathbf{0} \\ \lambda g(w_0^*, w^*) = 0 \end{cases}$$

$$\implies \begin{cases} \begin{pmatrix} 0 \\ \Sigma w \end{pmatrix} + \lambda \begin{pmatrix} -r \\ -m \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ \mathbb{1} \end{pmatrix} = \mathbf{0} \\ \lambda \left(\mu - w_0^* r - m^T w^* \right) = 0 \end{cases}$$

Now, we solve the system case-by-case.

Prepared by Max Shung

- **Case 1:** $\mu < w_0^* r + m^T w^*$
  Then we have $\lambda = 0$ and hence $0 - \lambda r + \gamma = 0 \implies \gamma = 0$.
  This implies that $\Sigma w^* = 0$ and by assumption $\Sigma \succ 0$ hence Σ^{-1} exists.
  Multiplying to the left on both sides gives $w^* = 0$, i.e. $w_0^* = 1$.
  So, All money is invested in bank account.
- **Case 2:** $\mu = w_0^* r + m^T w^*$
  In this case, $\lambda > 0$ hence the equation becomes

$$\begin{cases} 0 - \lambda r + \gamma 1 = 0 \\ \Sigma w^* - \lambda m + \gamma \mathbb{1} = \mathbf{0} \end{cases}$$

  From the first equation, $\gamma = \lambda r$.
  Putting into the second equation gives $\Sigma w^* = \lambda(m - r\mathbb{1})$.
  Multiplying both sides by Σ^{-1} yields

$$w^* = \lambda \Sigma^{-1}(m - r\mathbb{1})$$

  From $\mu = w_0^* r + m^T w^*$, we have

$$\mu = w_0^* r + \lambda m^T \Sigma^{-1}(m - r\mathbb{1})$$

  From $w_0 + \sum_{k=1}^{n} w_k = 1$, we have

$$1 - w_0^* = \mathbb{1}^T w^* = \lambda \mathbb{1}^T \Sigma^{-1}(m - r\mathbb{1})$$

  Solving the above equations, we can obtain

$$w_0^* = 1 - (\mu - r)\frac{\mathbb{1}^T \Sigma^{-1}(m - r\mathbb{1})}{(m - r\mathbb{1})^T \Sigma^{-1}(m - r\mathbb{1})}$$

  and

$$\lambda = \frac{\mu - r}{(m - r\mathbb{1})^T \Sigma^{-1}(m - r\mathbb{1})}$$

- Finally, putting λ back to above, we get

$$\begin{pmatrix} w_0^* \\ w^* \end{pmatrix} = (1 - \alpha) \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{all invested into bank account}} + \alpha \underbrace{\begin{pmatrix} 1 - \mathbb{1}^T \Sigma^{-1}(m - r\mathbb{1}) \\ \Sigma^{-1}(m - r\mathbb{1}) \end{pmatrix}}_{\substack{\text{Part of Money in bank account} \\ \text{Part of Money in stock market}}}$$

where $\alpha = \dfrac{\mu - r}{(m - r\mathbb{1})^T \Sigma^{-1}(m - r\mathbb{1})}$.

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— End of Lecture 25 —
— End of Class —